Existence of eigenvalues : Set up

- (Fundamental thm of algebra, existence of roots) :

Let $P(x) = C_n \cdot x^n + C_{n-1} \cdot x^{n-1} + \dots + C_0$ be a polynomial in $\mathbb{C}[x]$

Then, it has a root α s.t. $P(\alpha) = 0$. $\exists\, \alpha \in \mathbb{C}$ s.t. $P(\alpha) = 0$

$$\underbrace{P(x)}_{=0} = \underbrace{(x-\alpha) \cdot Q(x)}_{\text{at } x=\alpha} + \underbrace{R(x)}_{=0} = ?$$

$\underbrace{\deg R(x)}_{=0} < \underbrace{\deg (x-\alpha)}_{=1}$

Corollary: $P(x)$ can be factored i.e. $\exists\, \alpha_1, \dots, \alpha_n$ s.t.

$$P(x) = C_n \cdot (x-\alpha_n)(x-\alpha_{n-1}) \cdots (x-\alpha_1)$$

Existence of eigenvalues (over $\mathbb{C}$)

- Let V be a vector space over $\mathbb{C}$ with $\dim(V)=n$. Let $\varphi : V \to V$ be a linear operator. Then φ has an eigenval.

$\dim(V)=n$

Proof: Start with any $v \neq 0$

$$v, \varphi(v), \varphi^2(v), \ldots, \varphi^n(v) \text{ must be L.D.}$$

$$\exists\, c_0, \ldots c_n \in \mathbb{C} \text{ s.t. } c_n \cdot \varphi^n(v) + \cdots + c_1 \cdot \varphi(v) + c_0 \cdot v = 0$$

$$\text{Consider } P(x) = c_n \cdot x^n + \cdots + c_1 \cdot x + c_0$$

$$\exists\, \alpha_1, \ldots \alpha_n \text{ s.t. } P(x) = c_n (x-\alpha_n)(x-\alpha_{n-1}) \cdots (x-\alpha_1)$$

So
$$c_n \cdot \varphi^n(v) + \cdots + c_1 \cdot \varphi(v) + c_0 \cdot v$$
$$= c_n (\varphi - \alpha_n \cdot I) \cdots (\varphi - \alpha_1 \cdot I)(v)$$

Existence of eigenvalues (over $\mathbb{R}$)

- Let V be a finite-dimensional inner product space over $\mathbb{R}$ and let $\varphi : V \longrightarrow V$ be self-adjoint. Then φ has an eigenvalue.

<u>Proof:</u>

Define $V' = \{ u + iv \mid u, v \in V \}$

$\varphi' : V' \longrightarrow V'$ as

$$\varphi'(u + iv) = \varphi(u) + i \cdot \varphi(v)$$

Then $\varphi' : V' \longrightarrow V'$ has at least one eigenval. λ.

$$\varphi'(u + iv) = \lambda \cdot (u + i \cdot v)$$

$\varphi(u) = \lambda \cdot u$ and $\varphi(v) = \lambda \cdot v$

since $\varphi = \varphi^*$ $\lambda \in \mathbb{R}$

Rayleigh Quotients: Connecting algebra and optimization

For $\varphi : V \longrightarrow V$ self-adjoint, and $v \in V \setminus \{0_V\}$ define

$$R_\varphi(v) = \frac{\langle v, \varphi(v) \rangle}{\|v\|^2} = \left\langle \frac{v}{\|v\|}, \varphi\left(\frac{v}{\|v\|}\right) \right\rangle = R_\varphi\left(\frac{v}{\|v\|}\right)$$

- Let V be an i.p. space with $\dim(V) = n$. $\varphi : V \longrightarrow V$ self-adjoint with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$\lambda_1 = \max_{v \in V \setminus \{0_V\}} R_\varphi(v) \qquad \lambda_n = \min_{v \in V \setminus \{0_V\}} R_\varphi(v)$$

Proof:

For w_1 s.t. $\varphi(w_1) = \lambda_1 \cdot w_1$, $R_\varphi(w_1) = \lambda_1$

For $v = \sum c_i \cdot w_i$ with (say) $\sum |c_i|^2 = \|v\|^2 = 1$

$$R_\varphi(v) = \left\langle \sum c_i \cdot w_i , \sum c_j \cdot \lambda_j \cdot w_j \right\rangle = \sum |c_i|^2 \lambda_i \leq \underbrace{\sum |c_i|^2 \cdot \lambda_1}_{\leq \lambda_1}$$

$\therefore \forall v \quad R_\varphi(v) \leq \lambda_1$

Courant-Fischer Thm

- Let V be an i.p. space with $\dim(V) = n$. $\varphi : V \to V$ self-adjoint with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then $\forall k \in [n]$

$$\lambda_k = \max_{\substack{S \subseteq V \\ \dim(S)=k}} \ \min_{v \in S \setminus \{0_V\}} R_\varphi(v)$$

$$= \min_{\substack{S \subseteq V \\ \dim(S)=n-k+1}} \ \max_{v \in S \setminus \{0_V\}} R_\varphi(v)$$

Proof: Ex.

Positive (semi) definite operators

- A self-adjoint operator $\varphi : V \longrightarrow V$ is said to be
  - **Positive semidefinite** if $R_\varphi(v) \geq 0 \quad \forall\, v \neq 0_V$
  - **Positive definite** if $R_\varphi(v) > 0 \quad \forall\, v \neq 0_V$

e.g. Symmetric $A \in \mathbb{R}^{n \times n}$ is positive semidefinite if . . .

Characterizing Positive Semidefiniteness

- For self-adjoint $\varphi : V \longrightarrow V$, the following are equivalent

(1) $R_\varphi(v) \geq 0 \quad \forall v \neq 0_V$

(2) All eigenvalues of φ are non-negative

(3) $\exists$ linear transformation $\alpha : V \longrightarrow V$ s.t. $\varphi = \alpha^* \alpha$

Proof: (1) $\Rightarrow$ (2) $\quad \lambda_n = \min\limits_{v \neq 0} R_\varphi(v) \geq 0$

(2) $\Rightarrow$ (3) For eigenbasis $w_1 \dots w_n$
eigenvalues $\lambda_1 \dots \lambda_n \geq 0$

Consider $\alpha(w_i) = \sqrt{\lambda_i} \cdot w_i$

(3) $\Rightarrow$ (1) For any φ, $\langle v, \varphi(v) \rangle = \langle v, \alpha^* \alpha(v) \rangle$

$$= \langle \alpha(v), \alpha(v) \rangle$$

$$= \|\alpha(v)\|^2 \geq 0.$$